Quantum versus classical integrability in Ruijsenaars–Schneider systems

O Ragnisco[1] **and R Sasaki**[2]

[1] Department of Physics, Roma Tre University, Via Vasca Navale 84, Rome I-00146, Italy
[2] Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto 606-8502, Japan

Abstract
The relationship (resemblance and/or contrast) between quantum and classical integrability in Ruijsenaars–Schneider systems, which are one-parameter deformations of Calogero–Moser systems, is addressed. Many remarkable properties of classical Calogero and Sutherland systems (based on *any* root system) at equilibrium are reported in a previous paper (Corrigan–Sasaki). For example, the minimum energies, frequencies of small oscillations and the eigenvalues of Lax pair matrices at equilibrium are all 'integer valued'. In this paper we report that similar features and results hold for the Ruijsenaars–Schneider type of integrable systems based on the *classical* root systems.

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1. Introduction

If a many-body dynamical system is (Liouville) integrable at both classical and quantum levels, the classical system appears to share many 'quantum' features. For example, the frequencies of small oscillations near equilibrium are 'quantized' together with the eigenvalues of the associated Lax matrices at equilibrium. This phenomenon of close relation/contrast between quantum and classical integrability has been explored extensively by Corrigan–Sasaki [1] (this paper will be referred to as I hereafter) for Calogero–Moser (CM) systems [2–4] based on any root system [5–7]. In this paper we show that a similar phenomenon occurs in Ruijsenaars–Schneider (RS) [8–11] systems, which are one-parameter *deformations* of CM systems. The equations determining equilibrium can be presented in a form similar to *Bethe ansatz* equations. The equilibrium positions are described as the zeros of certain 'deformed' classical polynomials, which tend to the Hermite, Laguerre and Jacobi polynomials in the CM limit [1]. The frequencies of small oscillations at equilibrium are either integers or a sort of '*deformed*' integer according to whether the potential is rational or trigonometric, respectively. In the CM limits, these frequencies tend to those presented in I, which provides non-trivial support for the current results.

This paper is organized as follows. In section 2, the RS systems [8] are briefly introduced and the formulae for evaluating the frequencies of small oscillations at equilibrium are derived. In most cases, the evaluation of the frequencies is done numerically. In section 3, the rational RS systems with two types of confining potentials for the classical root systems A, B, C, BC and D [9, 10] are recapitulated. The formulae of the frequencies of the small oscillations are presented and compared with the corresponding results in Calogero systems [2] given in I. In section 4, the trigonometric RS systems for the classical root systems are reviewed briefly. The formulae of the frequencies of small oscillations at equilibrium are presented and compared with the results of the Sutherland systems [3] given in I.

2. Ruijsenaars-type systems

The RS systems are 'discrete' versions of the CM systems [2–4], that is the momentum variables appear in the Hamiltonian not as polynomials but as exponential (hyperbolic) functions. In a quantum theoretical setting this would mean that the operator (an 'analytic difference operator', according to Ruijsenaars) $e^{\pm\beta p} = e^{\mp i\beta\hbar\partial/\partial q}$ causes a shift of the wavefunction by a finite unit $\beta\hbar$ in the imaginary direction, i.e. $\psi(q)$ to $\psi(q \mp i\beta\hbar)$. The parameter β has the dimensions of a momentum^{-1} and can be expressed as $\beta = 1/mc$, in which m and c are constants of the dimensions of the *mass* and the *velocity*, respectively. The RS systems can also be considered as one-parameter (c) deformations of the CM systems (which correspond to the $c \to \infty$ limit) and are sometimes referred to, somewhat misleadingly, as 'relativistic' CM systems (see [12] for comments on this point).

The dynamical variables of the classical RS model are the coordinates $\{q_j | j = 1, \ldots, r\}$ and their canonically conjugate momenta $\{p_j | j = 1, \ldots, r\}$, with the Poisson bracket relations:

$$\{q_j, p_k\} = \delta_{jk} \qquad \{q_j, q_k\} = \{p_j, p_k\} = 0.$$

These will be denoted by vectors in ($\mathbb{R}^r$)

$$q = (q_1, \ldots, q_r) \qquad p = (p_1, \ldots, p_r)$$

in which r is the number of particles and also the rank of the underlying root system Δ. In this paper we will discuss those models associated with the *classical* root systems, namely A, B, C, D and BC. The fact that all the roots are neatly expressed in terms of the orthonormal basis of $\mathbb{R}^r$ makes formulation much simpler than those systems based on exceptional root systems. Throughout this paper we adopt the convention of $\beta \equiv 1$ for convenience.

Following Ruijsenaars–Schneider [8] and van Diejen [9], let us start with the following Hamiltonian:

$$H(p, q) = \sum_{j=1}^{r} \left[2\cosh p_j \sqrt{V_j(q)V_j^*(q)} - (V_j(q) + V_j^*(q)) \right] \tag{2.1}$$

in which V_j^* is the complex conjugate of V_j. The form of the function $V_j = V_j(q)$ is determined by the root system Δ as

$$A : V_j(q) = w(q_j) \prod_{k \neq j} v(q_j - q_k) \qquad j = 1, \ldots, r+1 \tag{2.2}$$

$$B, C, BC, D : V_j(q) = w(q_j) \prod_{k \neq j} v(q_j - q_k)v(q_j + q_k) \qquad j = 1, \ldots, r. \tag{2.3}$$

The elementary potential functions v and w depend on the nature of interactions (rational, trigonometric, etc) and the root system Δ.

2.1. Equilibrium position and frequencies of small oscillations

It is easy to see that the system has a stationary solution

$$p = 0 \qquad q = \bar{q} \tag{2.4}$$

of the canonical equations of motion:

$$\dot{q}_j = \frac{\partial H(p, q)}{\partial p_j} = \sum_{j=1}^{r} 2 \sin p_j \sqrt{V_j V_j^*} \qquad \dot{p}_j = -\frac{\partial H(p, q)}{\partial q_j} \tag{2.5}$$

in which $\bar{q}$ satisfies

$$\left. \frac{\partial H(0, q)}{\partial q_j} \right|_{\bar{q}} = 0 \qquad j = 1, \ldots, r. \tag{2.6}$$

By expanding the Hamiltonian around the stationary solution (2.4), we obtain

$$H(p, q) = K(p) + P(q) + \text{higher-order terms in } p, \tag{2.7}$$

in which the kinetic part K is quadratic in p

$$K(p) = \sum_{j=1}^{r} (p_j)^2 a_j \qquad a_j \equiv |V_j(\bar{q})| \tag{2.8}$$

and the 'potential' P is given by

$$P(q) = \sum_{j=1}^{r} \left(2\sqrt{V_j(q)V_j^*(q)} - (V_j(q) + V_j^*(q)) \right) = -\sum_{j=1}^{r} \left(\sqrt{V_j(q)} - \sqrt{V_j^*(q)} \right)^2. \tag{2.9}$$

This should be compared with the classical potential in CM systems, $V_C = \sum_{j=1}^{r} (\partial W / \partial q_j)^2$, in which W is the *prepotential* [7]. It is obvious that the equilibrium is achieved at the point(s) at which all the functions V_j become *real* and *positive*:

$$V_j(\bar{q}) = V_j(\bar{q})^* > 0 \qquad j = 1, 2, \ldots, r. \tag{2.10}$$

Therefore the 'minimal energy' $P(\bar{q})$ is always 0 in contrast to the CM cases.

Let us define the Hessian of the 'potential' P at equilibrium as B_{jk}:

$$B_{jk} \equiv \left. \frac{\partial^2 P(q)}{\partial q_j \partial q_k} \right|_{\bar{q}} \qquad j, k = 1, \ldots, r. \tag{2.11}$$

It is easy to verify that

$$B_{jk} = \frac{1}{2} \sum_{l=1}^{r} \frac{1}{a_l} \left. \frac{\partial IV_l}{\partial q_j} \right|_{\bar{q}} \left. \frac{\partial IV_l}{\partial q_k} \right|_{\bar{q}} \qquad IV_l(q) \equiv -i(V_l(q) - V_l^*(q)) \qquad j, k = 1, \ldots, r. \tag{2.12}$$

Thus the small oscillations near the stationary point (2.4) are described by the effective quadratic Hamiltonian in p and $q - \bar{q}$:

$$H_{\text{eff}}(p, q) = K(p) + \frac{1}{2} \sum_{j,k=1}^{r} B_{jk}(q - \bar{q})_j (q - \bar{q})_k. \tag{2.13}$$

In terms of a canonical transformation

$$p_j' = \sqrt{2a_j} \, p_j \qquad q_j' = \frac{1}{\sqrt{2a_j}} (q - \bar{q})_j \qquad j = 1, \ldots, r \tag{2.14}$$

the quadratic Hamiltonian reads

$$H_{\text{eff}}(p', q') = \frac{1}{2} \sum_{j=1}^{r} (p'_j)^2 + \frac{1}{2} \sum_{j,k=1}^{r} B'_{jk} q'_j q'_k \tag{2.15}$$

$$B'_{jk} \equiv (W^2)_{jk} \tag{2.16}$$

in which a real *symmetric*[3] $r \times r$ matrix W is defined by

$$W_{jk} \equiv \sqrt{a_j} \left. \frac{\partial I V_k}{\partial q_j} \right|_{\bar{q}} \frac{1}{\sqrt{a_k}}. \tag{2.17}$$

The frequencies of small oscillations at the equilibrium are given simply by the eigenvalues of a matrix $\tilde{W}$:

$$\tilde{W}_{jk} \equiv \left. \frac{\partial I V_k}{\partial q_j} \right|_{\bar{q}} \tag{2.18}$$

which are relatively easy to evaluate.

3. Ruijsenaars–Calogero systems

Here we will discuss the discrete analogue of the Calogero systems [2], to be called RC systems, which were introduced by van Diejen for classical root systems only [9, 10]. (For the definition of CM systems based on any root system and the associated Lax representation, etc, see [5–7].) The original Calogero systems [2] have a rational $(1/q^2)$ potential plus a harmonic (q^2) confining potential, having two coupling constants g and ω for the systems based on the simply-laced root systems, A and D, and three couplings ω and g_L for the long roots and g_S for the short roots in the B (C) root system. (For the rational Calogero systems, those based on B and C root systems are equivalent.)

The discrete Calogero systems have two varieties (deformations), according to the number of independent coupling constants. The first has two (three for the non-simply-laced root systems) coupling constants g (g_L and g_S) and a which corresponds to ω in the Calogero systems. This can be called a 'minimal' discretization of the Calogero systems. The second is introduced by van Diejen [9, 10] having three (four for the non-simply-laced root systems) coupling constants g (g_L and g_S) and a, b both of which correspond to ω. In this case, the B and C systems are not equivalent in contrast to the Calogero systems. The integrability (classical and quantum) of the latter was discussed by van Diejen in some detail [9, 10], whereas the former (the minimal discretization) is new and its integrability has not been discussed to the best of our knowledge. As we will show in the next subsection, the very orderly spectrum of the small oscillations would give strong evidence for its integrability. Another intuitive, convincing (and very probably true) argument is that the integrability of the former system is inherited from the latter in the limit $b \to \infty$.

3.1. Linear confining potential case

Let us first write down the explicit forms of the elementary potential functions v and w. For the simply-laced root systems A and D the elementary potential functions are

$$A, D : v(x) = 1 - ig/x \qquad w(x) = a + ix \tag{3.1}$$

in which a and g are real coupling constants assumed to be positive. For the non-simply-laced root systems B (C), BC, they are

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[3] This property stems from the structure of the functions V_j, (2.2), (2.3) and from the even nature of $v'(x)$.

$$B\ (C): v(x) = 1 - \mathrm{i} g_L/x \qquad w(x) = (a + \mathrm{i} x)(1 - \mathrm{i} g_S/x) \tag{3.2}$$

$$BC: v(x) = 1 - \mathrm{i} g_M/x \qquad w(x) = (a + \mathrm{i} x)(1 - \mathrm{i} g_S/x)(1 - \mathrm{i} g_L/2x) \tag{3.3}$$

in which g_L, g_M and g_S are the independent positive coupling constants for the long, middle and short roots, respectively. As in the Calogero case, those based on B and C systems are equivalent. In all these cases the 'potential' P (2.9) grows linearly in $|q|$ as $|q| \to \infty$. Except for the BC case, there are simple identities:

$$\sum_j \{V_j(q) + V_j(q)^*\} = \mathrm{const.} \tag{3.4}$$

Thus the Hamiltonian (2.1) can be replaced by a simpler one

$$H'(p, q) = 2 \sum_{j=1}^{r} \cosh p_j \sqrt{V_j(q) V_j^*(q)} \tag{3.5}$$

which is usually used as a starting point for the trigonometric (hyperbolic) interaction theory (see section 4). To be more precise, the identities are

$$A_r: \sum_{j=1}^{r+1} \{V_j(q) + V_j(q)^*\} = 2(r+1)a + r(r+1)g \tag{3.6}$$

and

$$B_r\ (C_r),\ D_r: \sum_{j=1}^{r} \{V_j(q) + V_j(q)^*\} = 2r(a + (r-1)g_L + g_S). \tag{3.7}$$

(For D_r, $g_L \to g$ and $g_S \equiv 0$.)

It is interesting to note that the equations determining the equilibrium (2.10), in general, can be cast in a form which looks similar to the *Bethe ansatz* equation. For example, for the elementary potential (3.1), (3.2), (2.10) read

$$A_r: \prod_{k \neq j} \frac{\bar{q}_j - \bar{q}_k - \mathrm{i} g}{\bar{q}_j - \bar{q}_k + \mathrm{i} g} = \frac{a - \mathrm{i}\bar{q}_j}{a + \mathrm{i}\bar{q}_j} \qquad j = 1, \ldots, r+1 \tag{3.8}$$

$$D_r: \prod_{k \neq j} \frac{\bar{q}_j - \bar{q}_k - \mathrm{i} g}{\bar{q}_j - \bar{q}_k + \mathrm{i} g} \frac{\bar{q}_j + \bar{q}_k - \mathrm{i} g}{\bar{q}_j + \bar{q}_k + \mathrm{i} g} = \frac{a - \mathrm{i}\bar{q}_j}{a + \mathrm{i}\bar{q}_j} \qquad j = 1, \ldots, r \tag{3.9}$$

$$B_r: \prod_{k \neq j} \frac{\bar{q}_j - \bar{q}_k - \mathrm{i} g_L}{\bar{q}_j - \bar{q}_k + \mathrm{i} g_L} \frac{\bar{q}_j + \bar{q}_k - \mathrm{i} g_L}{\bar{q}_j + \bar{q}_k + \mathrm{i} g_L} = \frac{a - \mathrm{i}\bar{q}_j}{a + \mathrm{i}\bar{q}_j} \frac{\bar{q}_j + \mathrm{i} g_S}{\bar{q}_j - \mathrm{i} g_S} \qquad j = 1, \ldots, r. \tag{3.10}$$

They determine the zeros of deformed Hermite and Laguerre polynomials. For A_r, let us define

$$\bar{q}_j = \sqrt{ag}\, y_j \qquad \delta = g/a \tag{3.11}$$

and introduce a degree $r+1$ polynomial

$$H_{r+1}^{(\delta)}(x) = 2^{r+1} \prod_{j=1}^{r+1} (x - y_j) \tag{3.12}$$

which is a deformation of the Hermite polynomial $H_{r+1}(x)$. For lower n, $H_n^{(\delta)}(x)$ are

$$H_n^{(\delta)}(x) = H_n(x) \qquad n = 0, 1, 2 \qquad H_3^{(\delta)}(x) = H_3(x) - 4x\delta$$

$$H_4^{(\delta)}(x) = H_4(x) - 32x^2\delta + 12\delta \qquad H_5^{(\delta)}(x) = H_5(x) - 160x^3\delta + 200x\delta + 48x\delta^2 \tag{3.13}$$

$$H_6^{(\delta)}(x) = H_6(x) - 640x^4\delta + x^2(736\delta^2 + 1680\delta) - 240\delta^2 - 360\delta \quad \ldots .$$

For B_r let us define

$$\bar{q}_j = \sqrt{ag_L}\, y_j \qquad \alpha = g_S/g_L - 1 \qquad \delta = g_L/a \qquad (3.14)$$

and introduce a degree r polynomial

$$L_r^{(\delta,\alpha)}(x) = (-1)^r \prod_{j=1}^{r} \left(x - y_j^2\right)/r! \qquad (3.15)$$

which is a deformation of the Laguerre polynomial $L_r^{(\alpha)}(x)$. For lower n, we have

$$L_n^{(\delta,\alpha)}(x) = L_r^{(\alpha)}(x) \qquad n = 0, 1 \qquad L_2^{(\delta,\alpha)}(x) = L_2^{(\alpha)}(x) - \frac{\delta}{2}(-2 + 3x - 3\alpha + 2x\alpha - \alpha^2)$$
$$(3.16)$$

$$\begin{aligned}
L_3^{(\delta,\alpha)}(x) = L_3^{(\alpha)}(x) - \frac{\delta}{6}(&-18 + 45x - 13x^2 - 33\alpha \\
&+ 42x\alpha - 6x^2\alpha - 18\alpha^2 + 9x\alpha^2 - 3\alpha^3 - 12\delta \\
&+ 22x\delta - 22\alpha\delta + 24x\alpha\delta - 12\alpha^2\delta + 6x\alpha^2\delta - 2\alpha^3\delta)
\end{aligned} \qquad (3.17)$$

$$\begin{aligned}
L_4^{(\delta,\alpha)}(x) = L_4^{(\alpha)}(x) - \frac{\delta}{24}(&-144 + 504x - 280x^2 + 34x^3 - 300\alpha \\
&+ 582x\alpha - 190x^2\alpha + 12x^3\alpha - 210\alpha^2 + 210x\alpha^2 - 30x^2\alpha^2 - 60\alpha^3 \\
&+ 24x\alpha^3 - 6\alpha^4 - 264\delta + 760x\delta - 241x^2\delta - 550\alpha\delta \\
&+ 950x\alpha\delta - 192x^2\alpha\delta - 385\alpha^2\delta + 366x\alpha^2\delta - 36x^2\alpha^2\delta - 110\alpha^3\delta \\
&+ 44x\alpha^3\delta - 11\alpha^4\delta - 144\delta^2 + 300x\delta^2 - 300\alpha\delta^2 + 420x\alpha\delta^2 - 210\alpha^2\delta^2 \\
&+ 180x\alpha^2\delta^2 - 60\alpha^3\delta^2 + 24x\alpha^3\delta^2 - 6\alpha^4\delta^2)\ldots.
\end{aligned} \qquad (3.18)$$

They are not the so-called '*q-deformed*' Hermite or Laguerre polynomials [13]. As in the Calogero systems, D_r is a special case $g_S = 0$ of the B_r theory described by $L^{(\delta,-1)}(x)$, which has a zero at $x = 0$ for all r (see (I.4.20)).

It is remarkable that the spectrum of the small oscillations at equilibrium is *completely independent* of the coupling constants g, a, g_L or g_S. In other words, the spectrum is the *topological invariant* of the theory. It is solely determined by the root system. In fact, the spectrum is

$$2(1 + e_j) \qquad j = 1, \ldots, r \qquad (3.19)$$

in which e_j is the jth *exponent* of the root system Δ. Explicitly, the spectrum is

$$A : 2 \times (1, 2, \ldots, r + 1) \qquad (3.20)$$

$$B\,(C),\, BC : 2 \times (2, 4, \ldots, 2r) \qquad (3.21)$$

$$D : 2 \times (2, 4, \ldots, 2r - 2, r). \qquad (3.22)$$

(The lowest frequency of the A series is due to the *centre of mass* motion which is not described by the data of the root system.) The situation is essentially the same as in the Calogero systems, in which the frequencies of the small oscillations are proportional to the above formula (3.19) and are independent of the coupling constant(s) of the rational potential [1]. We strongly believe that this very orderly spectrum is good evidence for the integrability of this type of model, as is the case for the Calogero systems.

3.2. Quadratic confining potential case

Let us first write down the explicit forms of the elementary potential functions v and w. For the simply-laced root systems A and D, the elementary potential functions are

$$v(x) = 1 - ig/x \qquad w(x) = (a + ix)(b + ix) \qquad a, b, g > 0 \qquad (3.23)$$

in which a and g are real coupling constants, assumed to be positive. For the non-simply-laced root systems B, C, BC the elementary potential functions are ($a, b, g_L, g_M, g_S > 0$):

$$B : v(x) = 1 - ig_L/x \qquad w(x) = (a + ix)(b + ix)(1 - ig_S/x) \qquad (3.24)$$

$$C : v(x) = 1 - ig_S/x \qquad w(x) = (a + 2ix)(b + 2ix)(1 - ig_L/2x) \qquad (3.25)$$

$$BC : v(x) = 1 - ig_M/x \qquad w(x) = (a + ix)(b + ix)(1 - ig_L/2x)(1 - ig_S/x). \qquad (3.26)$$

The C system is slightly different from the one given by van Diejen [10], since the latter is a quantum theory. In the limit $\hbar \to 0$, (3.25) is the same as van Diejen's. Here again those based on B and C systems are equivalent in terms of the overall scaling of the potential and scaling of the coupling constants. This fact is reflected in their spectra (3.29), (3.30). In all these cases the 'potential' P (2.9) grows quadratically in $|q|$ as $|q| \to \infty$. In the present case the identities (3.4) are replaced by

$$\sum_j \{V_j(q) + V_j(q)^*\} = -2q^2 + \text{const} \qquad (3.27)$$

except for the BC case.

Again the equations determining the equilibrium (2.10) are *Bethe ansatz*-like equations which are expected to determine the two-parameter deformation of the Hermite and Laguerre polynomials. These will be discussed elsewhere. The spectrum of the small oscillations at equilibrium has a very simple form. Explicitly, the spectrum is

$$A : 2j[a + b + g(r - (j - 1)/2)] \qquad j = 1, 2, \ldots, r + 1 \qquad (3.28)$$

$$B : 4j[a + b + g_S + g_L(2r - j - 1)] \qquad j = 1, 2, \ldots, r \qquad (3.29)$$

$$C : 8j[a + b + g_L + 2g_S(2r - j - 1)] \qquad j = 1, 2, \ldots, r \qquad (3.30)$$

$$BC : 4j[a + b + g_L/2 + g_S + g_M(2r - j - 1)] \qquad j = 1, 2, \ldots, r \qquad (3.31)$$

$$D : 4j[a + b + g(2r - j - 1)] \qquad j = 1, 2, \ldots, r - 1 \qquad \text{and} \quad 2r(a + b + g(r - 1)). \qquad (3.32)$$

The Calogero models are obtained in the singular limit, $a, b \to \infty$ and by division by ab. In this limit, the above spectrum of small oscillations, (3.28)–(3.32), will be proportional to those of the Calogero models, i.e. (3.19), (3.20)–(3.22), as expected. The simple spectra of the linear confining potential case (3.20)–(3.22) are obtained by dividing (3.28)–(3.32) by b and taking the limit $b \to \infty$.

It is interesting to compare the above spectrum of small oscillations with the *quantum* energy eigenvalues. The quantum spectrum is given by van Diejen [9]:

$$A : E_{\bar{n}} = \sum_{1 \leqslant j \leqslant r+1} n_j[n_j + 2(a + b) - 1 + 2g(r + 1 - j)] \qquad (3.33)$$

$$B : E_{\bar{n}} = 4 \sum_{1 \leqslant j \leqslant r} n_j[n_j + a + b - 1 + g_S + 2g_L(r - j)] \qquad (3.34)$$

$$C : E_{\vec{n}} = 8 \sum_{1 \leqslant j \leqslant r} n_j [n_j + a + b - 1 + g_L + 4g_S(r - j)] \tag{3.35}$$

$$BC : E_{\vec{n}} = 4 \sum_{1 \leqslant j \leqslant r} n_j [n_j + a + b - 1 + g_L/2 + g_S + 2g_M(r - j)] \tag{3.36}$$

in which $\vec{n} = (n_1, \ldots, n_r, (n_{r+1}))$ is a set of 'quantum numbers' parametrizing the eigenstates. They are non-increasing, non-negative integers $(n_1 \geqslant n_2 \geqslant \cdots n_r \geqslant (n_{r+1}) \geqslant 0)$. In fact, the B and C formulae are special cases of the BC formula. However, the D formula is yet to be derived. The $r\,(r+1)$ independent 'lowest lying' modes corresponding to the quantum numbers

$$\vec{n} = (1, 0, \ldots, 0), (1, 1, 0, \ldots, 0), \ldots, (1, 1, \ldots, 1, 0), (1, 1, \ldots, 1) \tag{3.37}$$

have energies,

$$A : 2j[a + b + g(r - (j - 1)/2)] \qquad j = 1, \ldots, r + 1 \tag{3.38}$$

$$B : 4j[a + b + g_S + g_L(2r - 1 - j)] \qquad j = 1, \ldots, r \tag{3.39}$$

$$C : 8j[a + b + g_L + 2g_S(2r - 1 - j)] \qquad j = 1, \ldots, r \tag{3.40}$$

$$BC : 4j[a + b + g_L/2 + g_S + g_M(2r - 1 - j)] \qquad j = 1, \ldots, r \tag{3.41}$$

which are exactly the same as (3.28)–(3.31).

4. Ruijsenaars–Sutherland systems

The discrete analogue of the Sutherland systems [3], to be called RS systems, was introduced originally by Ruijsenaars and Schneider [8] for the A-type root system. The quantum eigenfunctions of the A-type RS systems are called Macdonald polynomials [15], which are a one-parameter deformation of the Jack polynomials [16]. Here we will discuss RS systems for all the classical root systems, A, B, C, D and BC [10]. The structure of the functions $\{V_j(q)\}$, (2.2) and (2.3) are the same as in the RC systems, but the elementary potential functions v and w are trigonometric instead of rational. Because of the identity $\sum_{j=1}^{r} \{V_j(q) + V_j(q)^*\} = \text{const}$, (3.4), the Hamiltonian (2.1) can be replaced by a simpler one

$$H'(p, q) = 2 \sum_{j=1}^{r} \cosh p_j \sqrt{V_j(q) V_j^*(q)} \tag{4.1}$$

which is obviously positive definite. This is also valid for the BC case in contrast to the rational potential cases discussed in the preceding section 3.

The elementary potential functions v and w are

$$A, D : v(x) = \cosh \gamma - i \sinh \gamma \cot x \qquad w(x) \equiv 1 \tag{4.2}$$

$$B : v(x) = \cosh \gamma_L - i \sinh \gamma_L \cot x \qquad w(x) = \cosh \gamma_S - i \sinh \gamma_S \cot x \tag{4.3}$$

$$C : v(x) = \cosh \gamma_S - i \sinh \gamma_S \cot x \qquad w(x) = \cosh \gamma_L - i \sinh \gamma_L \cot 2x \tag{4.4}$$

$$C' : v(x) = \cosh \gamma_S - i \sinh \gamma_S \cot x \qquad w(x) = (\cosh \gamma_L - i \sinh \gamma_L \cot 2x)^2 \tag{4.5}$$

$$BC : v(x) = \cosh \gamma_M - i \sinh \gamma_M \cot x$$

$$w(x) = (\cosh \gamma_S - i \sinh \gamma_S \cot x)(\cosh \gamma_L - i \sinh \gamma_L \cot 2x) \tag{4.6}$$

in which γ_L, γ_M and γ_S are the positive coupling constants for the long, middle and short roots, respectively. C, C' and BC cases are special cases of the most general integrable interactions including the long roots $(\alpha_L^2 = 4)$ introduced by van Diejen [11]. Note that in our paper only the classical dynamics $\hbar \to 0$ is discussed. That is, van Diejen's constant $\gamma = i\beta\hbar/2$ (equation (2.7) in [11]) is treated as vanishing.

For the systems based on an A-type root system, the above identity (3.4)

$$A_r : \sum_{j=1}^{r+1} V_j(q) = \sinh[(r+1)\gamma]/\sinh\gamma \tag{4.7}$$

is known in a different context [14]. For the other root systems the identity (3.4) reads

$$\sum_{j=1}^{r} \{V_j(q) + V_j(q)^*\} = \begin{cases} 2\sinh[r\gamma_L]\cosh[(r-1)\gamma_L + \gamma_S]/\sinh\gamma_L & B_r \\ 2\sinh[r\gamma_S]\cosh[(r-1)\gamma_S + \gamma_L]/\sinh\gamma_S & C_r \\ \sinh[(2r-1)\gamma]/\sinh\gamma + 1 & D_r. \end{cases} \tag{4.8}$$

It is easy to verify that the B_r formula reduces to the D_r one for $\gamma_L \to \gamma$ and $\gamma_S = 0$. Had we started from the simplified Hamiltonian (4.1) instead of the original one (2.1), the above constants (4.7)–(4.8) would give the minimal energies. In the simply-laced A and D cases, the rhs of (4.7) and (4.8) can be interpreted as the '*q-deformed integer*' version of the dimensionality of A_r and D_r vector representations, $[r+1]_q$ and $1 + [2r-1]_q$.

The equilibrium of the A-type theories is achieved at '*equally-spaced*'

$$\bar{q} = \pi(0, 1, \ldots, r-1, r)/(r+1) + \xi(1, 1, \ldots, 1) \qquad \xi \in \mathbb{R} : \text{arbitrary} \tag{4.9}$$

configuration, as in the original Sutherland systems. In all the other cases, the equilibrium and the frequencies of the small oscillations are determined by solving the *Bethe ansatz*-like equations (2.10) numerically. Certain one-parameter deformation of the Jacobi polynomials [1] is expected to describe the equilibrium for B, C, D and BC systems, which will be discussed elsewhere. The spectrum of the small oscillations at equilibrium has a very simple form. Explicitly, the spectrum is

$$A : 4\sinh[(r+1-j)\gamma]\sinh[j\gamma]/\sinh\gamma \qquad j = 1, 2, \ldots, r+1 \tag{4.10}$$

$$B : 4\sinh[(2r-1-j)\gamma_L + \gamma_S]\sinh[j\gamma_L]/\sinh\gamma_L \qquad j = 1, 2, \ldots, r-1$$
$$2\sinh[(r-1)\gamma_L + \gamma_S]\sinh[r\gamma_L]/\sinh\gamma_L \tag{4.11}$$

$$C : 4\sinh[(2r-1-j)\gamma_S + \gamma_L]\sinh[j\gamma_S]/\sinh\gamma_S \qquad j = 1, 2, \ldots, r \tag{4.12}$$

$$C' : 4\sinh[(2r-1-j)\gamma_S + 2\gamma_L]\sinh[j\gamma_S]/\sinh\gamma_S \qquad j = 1, 2, \ldots, r \tag{4.13}$$

$$BC : 4\sinh[(2r-1-j)\gamma_M + 2\gamma_L + \gamma_S]\sinh[j\gamma_M]/\sinh\gamma_M \qquad j = 1, 2, \ldots, r \tag{4.14}$$

$$D : 4\sinh[(2r-1-j)\gamma]\sinh[j\gamma]/\sinh\gamma \qquad j = 1, 2, \ldots, r-2$$
$$2\sinh[(r-1)\gamma]\sinh[r\gamma]/\sinh\gamma \qquad \text{twofold degenerate.} \tag{4.15}$$

The spectrum of the A system (4.10)[4] reflects the symmetry of the Dynkin diagram $j \leftrightarrow r+1-j$. The twofold degeneracy of the D spectrum (4.15) also reflects the symmetry of the D Dynkin diagram.

The original Sutherland models are obtained in the singular limit in which all the coupling constant(s) become infinitesimally small: $0 < \gamma, \gamma_L, \gamma_M, \gamma_S \ll 1$. In this limit, the spectrum of small oscillations at equilibrium will be linear in the coupling constant(s). In these limits,

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[4] This formula was known to Ruijsenaars [17].

the known spectrum of small oscillations obtained in I is reproduced. As for A_r, the spectrum (4.10) becomes

$$A : 4(r + 1 - j)j\gamma \qquad j = 1, \ldots, r + 1 \qquad\qquad (4.16)$$

which is equation (5.16) of Corrigan–Sasaki [1], to be referred to as (I.5.16). For B_r, the spectrum (4.11) in the limit is

$$B_r : 4[(2r - 1 - j)\gamma_L + \gamma_S]j \qquad j = 1, \ldots, r - 1 \qquad 2[(r - 1)\gamma_L + \gamma_S]r \quad [2] \quad (4.17)$$

which is (I.5.74). For C_r', the spectrum (4.13) in the limit is

$$C' : 4[(2r - 1 - j)\gamma_S + 2\gamma_L]j \qquad j = 1, \ldots, r \qquad\qquad (4.18)$$

which is (I.5.81). For D_r, the spectrum (4.15) in the limit is

$$D : 4(2r - 1 - j)j\gamma \qquad j = 1, \ldots, r - 2 \qquad 2r(r - 1)\gamma \quad [2] \quad (4.19)$$

which is (I.5.60). The limiting spectra provide non-trivial supporting evidence for the formulae (4.10)–(4.15).

It is a challenge to understand the connection between the quantum spectrum of the Ruijsenaars–Sutherland systems, equation (5.17) of [15] and the above spectrum of small oscillations (4.10)–(4.15). It would be interesting to evaluate the eigenvalues of the Lax matrices at equilibrium as shown for the CM systems [1]. However, the knowledge of the Lax pairs for the RS systems is still quite limited [18].

Acknowledgments

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